Kor: The hard margin SVM has a unique solution.

Proof: The hard margin SVM is given by

$$\min_{w \in \mathbb{R}^n} \underbrace{\tfrac{1}{2}\|w\|^2}_{=: f_0(w)} \quad \text{subject to} \quad \begin{cases} \forall i = 1 \dots m \\ \underbrace{y^{(i)} w \cdot x^{(i)} \geq 1}_{\Leftrightarrow\ f_i(w) \leq 0} \end{cases}$$

for $f_i(w) = y^{(i)} w x^{(i)} - 1$

Now $\partial_w f_0(w) = w$ (gradient)

$\partial_w^2 f_0(w) = \mathbb{1}$ (hessian)

$\Rightarrow$ f_0 is strictly convex

Furthermore, f_0 is coercive, $\mathbb{R}^n$ closed and f_i for $1 \leq i \leq m$ are affine $\Rightarrow$ convex

$\Rightarrow$ Thms above state

$\exists\, x^*$ local min

and this is unique. $\square$